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Polynomial states for SU(3) and SO(5) in a Demazure–Tits basis

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Abstract. We construct basis states for SU(3) and for SO(5) that are polynomials in the states of the fundamental representations; they are reduced according to the finite Demazure–Tits subgroup, which acts on basis states in the same manner that the Weyl group acts on weights.

1. Introduction

The Demazure–Tits (DT) group (Demazure and Groethendieck 1963, 1964, Tits 1966, Michel *et al* 1988) of a simple compact Lie group is an extension of its Weyl group; whereas the Weyl group acts on weights, the DT group acts on basis states and is a subgroup of the Lie group in question.

When an element of the DT group acts on a state its weight is Weyl-reflected; accordingly Weyl orbits are not mixed and orbit labels can be used as state labels; we will see below how additional labels can be introduced to remove the remaining degeneracy.

In section 2 we derive basis states for SU(3), reduced according to DT; they are polynomials in the basis states of the (1,0) and (0,1) (fundamental) representations. In section 3 the same task is carried out for SO(5). Section 4 contains some concluding remarks, including a discussion on the use of the states to compute generator matrix elements in a DT basis.

2. SU(3) $\supset$ DT basis states

We construct basis states of the irrep (p,q) of SU(3) as polynomials of degree p in η_1, η_2, η_3, the basis states of the irrep (1,0), and of degree q in $\eta_1^*, \eta_2^*, \eta_3^*$, the basis states of the irrep (0,1). These fundamental states are shown in figure 1.

Since Demazure–Tits transformations do not change the (Weyl) orbit it is convenient to begin with the SU(3) orbit-generating function (Michel *et al* 1988):

$$\begin{aligned} F(P,Q;A,B) = (1-PQ)^{-2}\{&[(1-P^3)(1-P^2B)(1-PA)]^{-1} \\ &+[(1-P^2B)(1-PA)(1-QB)]^{-1}QB \\ &+[(1-PA)(1-QB)(1-Q^2A)]^{-1}Q^2A \\ &+[(1-QB)(1-Q^2A)(1-Q^3)]^{-1}Q^3\} \end{aligned} \tag{2.1}$$

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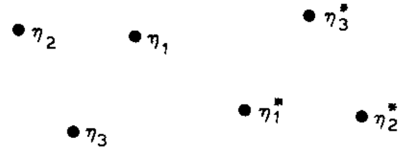

Figure 1. States of the fundamental irreps of SU(3).

In the power expansion of F the coefficient of $P^pQ^qA^aB^b$ is the multiplicity of the orbit $[a,b]$ in the irrep (p,q); the orbit labels a,b are the (non-negative) components of the highest weight of the orbit. We call the eight terms P^3, P^2B etc in (2.1) elementary orbits; they form an integrity basis in the sense that the general orbit is represented by a product of powers of them.

The generating function (2.1) not only counts orbits but implicitly instructs how to construct the relevant states. Before giving explicit expressions for the elementary orbits (more precisely for their highest states) we must examine the SU(3) character or weight generating function (Patera and Sharp 1979)

$$\begin{aligned} G(P,Q;A,B) = {} & [(1-PA)(1-PB^{-1})(1-QB)(1-QA^{-1})]^{-1} \\ & \times \{(1-PA^{-1}B)^{-1} + (1-QAB^{-1})^{-1}QAB^{-1}\}. \end{aligned} \tag{2.2}$$

The correspondence with actual states is given by $PA \sim \eta_1$, $PB^{-1} \sim \eta_3$, $QB \sim \eta_3^*$, $QA^{-1} \sim \eta_1^*, PA^{-1}B \sim \eta_2, QAB^{-1} \sim \eta_2^*$. The orbit-generating function (2.1) is the non-negative degree part, in A and B, of (2.2). The expansion of (2.2) consists of products of powers of $\eta_1, \eta_2, \eta_3, \eta_1^*, \eta_2^*, \eta_3^*$, with $\eta_2\eta_2^*$ never appearing together; such products of powers of the states of the fundamental irreps constitute a consistent listing of the states of all irreps (p,q) of SU(3) with p and q the degrees in the unstarred and starred variables respectively. $\eta_2\eta_2^*$ never appear together because of the 'unwanted' scalar $B = \eta_1\eta_1^* + \eta_2\eta_2^* + \eta_3\eta_3^*$; any state containing it as a factor would belong to an irrep lower than indicated by its degrees. We formally set $B=0$ and replace $\eta_2\eta_2^*$ by $-\eta_1\eta_1^* - \eta_3\eta_3^*$ wherever it appears.

The explicit form of the elementary orbits now may be taken as follows:

$$\begin{array}{lll} P^3 \sim \eta_1\eta_2\eta_3 & P^2B \sim \eta_1\eta_2 & PA \sim \eta_1 \\ QB \sim \eta_3^* & Q^2A \sim \eta_2^*\eta_3^* & Q^3 \sim \eta_1^*\eta_2^*\eta_3^* \\ (PQ)_1 \sim \eta_3\eta_3^* - \omega^2\eta_1\eta_1^* & (PQ)_2 \sim -\eta_3\eta_3^* + \omega\eta_1\eta_1^* & \end{array} \tag{2.3}$$

where ω is the primitive cube root of unity $\exp(2\pi i/3)$.

Until further notice we ignore the two 'centre orbits' $(PQ)_1, (PQ)_2$ (they occur at the centre of the octet irrep (1,1)). At the end powers of them will be coupled to orbits obtained from the other six. The variable η_i now never appears multiplied by η_i^*, so it is convenient to write $\eta_i^* = \eta_i^{-1}$ (η_i and η_i^* have opposite weights).

It is easy to verify that a term $P^pQ^qA^aB^b$ appears no more than once in the expansion of $(1-PQ)^2F(P,Q;A,B)$; orbit multiplicities greater than one arise later from coupling to powers of the centre orbits $(PQ)_1, (PQ)_2$.

The DT subgroup of SU(3) happens to be isomorphic to O, the subgroup of SO(3) that leaves the cube invariant (for a discussion of O, see for example Lomont 1959).

We obtain standard matrices for the generating elements R_1, R_2 for each irrep of DT by the use of prototype basis states. For the defining irrep, Γ_4, we use η_1, η_2, η_3, as prototype states with phase conventions as in Michel *et al* (1988). Then

$$R_1 = \begin{pmatrix} 0 & \bar{1} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \bar{1} \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.4}$$

For the scalar irrep Γ_1 we have

$$R_1 = R_2 = 1. \tag{2.5}$$

For the one-dimensional irrep Γ_2 we use the prototype state $\eta_1\eta_2\eta_3$. Then

$$R_1 = R_2 = -1. \tag{2.6}$$

For the two-dimensional irrep Γ_3 we choose the prototype basis states as, respectively $(PQ)_1$ and $(PQ)_2$ of (2.3). Then

$$R_1 = \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.7}$$

Finally, for Γ_5, we take prototype states as, respectively, $\eta_2\eta_3, \eta_3\eta_1, \eta_1\eta_2$. Then

$$R_1 = \begin{pmatrix} 0 & 1 & 0 \\ \bar{1} & 0 & 0 \\ 0 & 0 & \bar{1} \end{pmatrix} \qquad R_2 = \begin{pmatrix} \bar{1} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \bar{1} & 0 \end{pmatrix}. \tag{2.8}$$

At this stage the general state is $|rst\rangle = \eta_1^r\eta_2^s\eta_3^t$, with weight $\{a,b\} = \{r-s, s-t\}$. The dominant state of an orbit, i.e., the one described by the right-hand side of (2.1), has $r \geq s \geq t$; its weight $[a,b]$ labels the orbit. The (later) coupling to the centre orbits $(PQ)_1$, $(PQ)_2$ changes neither the orbit (or weight) labels nor the difference $p-q$; hence we may write

$$\begin{aligned} r &= \tfrac{1}{3}(p-q+2a+b) \\ s &= \tfrac{1}{3}(p-q-a+b) \\ t &= \tfrac{1}{3}(p-q-a-2b). \end{aligned} \tag{2.9}$$

The six states of the orbit (we assume $r > s > t$) are

$$\begin{aligned} &|1\rangle = |rst\rangle \qquad |2\rangle = |str\rangle \qquad |3\rangle = |trs\rangle \\ &|4\rangle = |srt\rangle \qquad |5\rangle = |rts\rangle \qquad |6\rangle = |tsr\rangle. \end{aligned} \tag{2.10}$$

The DT irreps contained in the orbit may be found with the help of the DT character table (Lomont 1959, Michel *et al* 1988) and the character of the orbit. The action of the generating elements on the orbit states is

$$\begin{aligned} R_1|xyz\rangle &= (-1)^y|yxz\rangle \\ R_2|xyz\rangle &= (-1)^z|xzy\rangle. \end{aligned} \tag{2.11}$$

We must consider four cases.

Case 1. a, b both even; r, s, t have the same parity. The DT content of the orbit is $\Gamma_1 \oplus \Gamma_2 \oplus 2\Gamma_3$.

From $R_1|{}^{rst}_{\Gamma_1}\rangle = R_2|{}^{rst}_{\Gamma_1}\rangle = |{}^{rst}_{\Gamma_1}\rangle$ we find the Γ_1 state

$$\left|\begin{matrix} rst \\ \Gamma_1 \end{matrix}\right\rangle = |1\rangle + |2\rangle + |3\rangle + \lambda|4\rangle + \lambda|5\rangle + \lambda|6\rangle \tag{2.12}$$

with $\lambda = (-1)^r = (-1)^s = (-1)^t$.

From $R_1|{}^{rst}_{\Gamma_2}\rangle = R_2|{}^{rst}_{\Gamma_2}\rangle = -|{}^{rst}_{\Gamma_2}\rangle$ we find the Γ_2 state

$$\left|\begin{matrix} rst \\ \Gamma_2 \end{matrix}\right\rangle = |1\rangle + |2\rangle + |3\rangle - \lambda|4\rangle - \lambda|5\rangle - \lambda|6\rangle. \tag{2.13}$$

From $R_1 R_2|{}^{rst}_{i\Gamma_3 1}\rangle = \omega^2|{}^{rst}_{i\Gamma_3 1}\rangle$ we find two states,

$$\left|\begin{matrix} rst \\ 1\Gamma_3 1 \end{matrix}\right\rangle = |1\rangle + \omega^2|2\rangle + \omega|3\rangle \qquad \left|\begin{matrix} rst \\ 2\Gamma_3 1 \end{matrix}\right\rangle = |4\rangle + \omega^2|5\rangle + \omega|6\rangle \tag{2.14a}$$

the notation$|{}^{rst}_{i\Gamma_3 k}\rangle$ denotes the kth component of the ith copy of the Γ_3 irrep in the orbit coresponding to rst. The two second components are found by applying R_2 to the first components:

$$\left|\begin{matrix} rst \\ 1\Gamma_3 2 \end{matrix}\right\rangle = \lambda|5\rangle + \lambda\omega^2|4\rangle + \lambda\omega|6\rangle \qquad \left|\begin{matrix} rst \\ 2\Gamma_2 2 \end{matrix}\right\rangle = \lambda|2\rangle + \lambda\omega^2|1\rangle + \lambda\omega|3\rangle. \tag{2.14b}$$

Case 2. a, b both odd; the parity of s is opposite to that of r and t. The DT content of the orbit is $\Gamma_4 \oplus \Gamma_5$. The first component of the Γ_4 irrep is found from $R_2|{}^{rst}_{\Gamma_4 1}\rangle = |{}^{rst}_{\Gamma_4 1}\rangle$. The other components are then obtained from $|{}^{rst}_{\Gamma_4 2}\rangle = R_1|{}^{rst}_{\Gamma_4 1}\rangle$ and $|{}^{rst}_{\Gamma_4 3}\rangle = R_2|{}^{rst}_{\Gamma_4 2}\rangle$. We find

$$\left|\begin{matrix} rst \\ \Gamma_4 1 \end{matrix}\right\rangle = |2\rangle + \lambda|4\rangle \qquad \left|\begin{matrix} rst \\ \Gamma_4 2 \end{matrix}\right\rangle = |1\rangle + \lambda|6\rangle \qquad \left|\begin{matrix} rst \\ \Gamma_4 3 \end{matrix}\right\rangle = |3\rangle + \lambda|5\rangle. \tag{2.15}$$

where $\lambda = (-1)^r = (-1)^t = -(-1)^s$.

Similarly

$$\left|\begin{matrix} rst \\ \Gamma_5 1 \end{matrix}\right\rangle = |2\rangle - \lambda|4\rangle \qquad \left|\begin{matrix} rst \\ \Gamma_5 2 \end{matrix}\right\rangle = |1\rangle - \lambda|6\rangle \qquad |{}^{rst}_{\Gamma_5 3}\rangle = |3\rangle - \lambda|5\rangle. \tag{2.16}$$

Case 3. a even, b odd; t's parity is opposite to that of r and s. The DT content is $\Gamma_4 \oplus \Gamma_5$.

We find

$$\left|\begin{matrix} rst \\ \Gamma_4 1 \end{matrix}\right\rangle = |3\rangle + \lambda|6\rangle \qquad \left|\begin{matrix} rst \\ \Gamma_4 2 \end{matrix}\right\rangle = |2\rangle + \lambda|5\rangle \qquad \left|\begin{matrix} rst \\ \Gamma_4 3 \end{matrix}\right\rangle = |1\rangle + \lambda|4\rangle \tag{2.17}$$

with $\lambda = (-1)^r = (-1)^s = -(-1)^t$, and

$$\left|\begin{matrix} rst \\ \Gamma_5 1 \end{matrix}\right\rangle = |3\rangle - \lambda|6\rangle \qquad \left|\begin{matrix} rst \\ \Gamma_5 2 \end{matrix}\right\rangle = |2\rangle - \lambda|5\rangle \qquad \left|\begin{matrix} rst \\ \Gamma_5 3 \end{matrix}\right\rangle = |1\rangle - \lambda|4\rangle \tag{2.18}$$

Case 4. a odd b even; the parity of r is opposite to that of s and t. The DT content is $\Gamma_4 \oplus \Gamma_5$. We find

$$\left|\begin{matrix} rst \\ \Gamma_4 1 \end{matrix}\right\rangle = |1\rangle - \lambda|5\rangle \qquad \left|\begin{matrix} rst \\ \Gamma_4 2 \end{matrix}\right\rangle = |3\rangle - \lambda|4\rangle \qquad \left|\begin{matrix} rst \\ \Gamma_4 3 \end{matrix}\right\rangle = |2\rangle - \lambda|6\rangle \tag{2.19}$$

with $\lambda = (-1)^r = -(-1)^s = -(-1)^t$, and

$$\left|\begin{matrix} rst \\ \Gamma_5 1 \end{matrix}\right\rangle = |1\rangle + \lambda|5\rangle \qquad \left|\begin{matrix} rst \\ \Gamma_5 2 \end{matrix}\right\rangle = |3\rangle + \lambda|4\rangle \qquad \left|\begin{matrix} rst \\ \Gamma_5 3 \end{matrix}\right\rangle = |2\rangle + \lambda|6\rangle. \tag{2.20}$$

We find the states of a triangular orbit by treating it as a special case of a hexagonal orbit. Both types of triangular orbit are handled simultaneously by letting the state $\eta_1^r \eta_2^r \eta_3^t$ represent the orbit with $r > t$ for a $[0, b]$ orbit and $r < t$ for an $[a, 0]$ orbit. The states $|4\rangle, |5\rangle, |6\rangle$ of the hexagonal orbit become equal to the states $|1\rangle, |2\rangle, |3\rangle$, respectively.

For case 1 (b even) we see from (2.11)–(2.13) that for r even $\lambda = 1$ and the Γ_1 state survives while the Γ_2 state vanishes; for r odd $\lambda = -1$ and the Γ_2 state survives while the Γ_1 state vanishes. In both cases the two Γ_3 irreps become identical so the second should be ignored.

For case 3 (b odd) we see from (2.17) and (2.18) that the Γ_4 irrep survives and the Γ_5 irrep vanishes for r even, the reverse for r odd.

The point orbit ($a = b = 0$) has $r = s = t$. The three states of the triangular orbit become identical. Then the Γ_3 states vanish; only the Γ_1 state survives if r is even, only the Γ_2 state if r is odd, not a surprising result since the state is $(\eta_1\eta_2\eta_3)^r$.

We call the states $\left|\begin{smallmatrix} rst \\ i\Gamma_j k \end{smallmatrix}\right\rangle$, developed up to now, 'orbit states'; i labels a copy of the DT irrep Γ_j, and k labels its component in the orbit $[a, b] = [r - s, s - t]$, $[t - r, 0]$ when $t > r$ and $s = r$. Although they are mutually orthogonal and could easily be normalized, we have not bothered to do so, because the coupling to the centre orbits, to be accomplished next, would spoil the normalization and introduce non-orthogonality. The orthogonalization would then be impossible except say by a Schmidt procedure or numerical diagonalization of a metric matrix. In section 4 we argue that for most practical purposes, for example determination of generator matrix elements, there is little to be gained by orthonormalization.

The centre states $(PQ)_1$ and $(PQ)_2$ belong to the point orbit in the SU(3) irrep (1,1); they span the DT (or O) irrep Γ_3. The orbit states developed earlier in this section must now be coupled to powers of them. Patera *et al* (1978) have given generating functions for polynomial tensors in the components of Γ_3 (as well as of other tensors). They are

$$B(\Gamma_1, \Gamma_3; \lambda) = [(1 - \lambda^2)(1 - \lambda^3)]^{-1} \tag{2.21a}$$

$$B(\Gamma_2, \Gamma_3; \lambda) = [(1 - \lambda^2)(1 - \lambda^3)]^{-1}\lambda^3 \tag{2.21b}$$

$$B(\Gamma_3, \Gamma_3; \lambda) = [(1 - \lambda^2)(1 - \lambda^3)]^{-1}(\lambda + \lambda^2). \tag{2.21c}$$

Equation (2.21a) tells us that there are two elementary scalars (Γ_1 tensors) of degrees 2 and 3 and that all scalars are products of powers of them. Equation (2.21b) says there is one elementary Γ_2 tensor, of degree 3, and other Γ_2 tensors are obtained by multiplying by products of powers of the elementary scalars. Equation (2.21c) informs us there are two elementary Γ_3 tensors of degrees 1 and 2 and that others are obtained by multiplying by products of powers of scalars.

The elementary scalars are easily found to be $(PQ)_1(PQ)_2$ (degree 2) and $(PQ)_1^3 + (PQ)_2^3$ (degree 3). The elementary Γ_2 tensor is $(PQ)_1^3 - (PQ)_2^3$. The elementary Γ_3 tensors are $\binom{(PQ)_1}{(PQ)_2}$ and $\binom{(PQ)_2^2}{(PQ)_1^2}$. We denote centre states by $|{}^{cd}_{g\Gamma_h i}\rangle$; c and d are the powers of the quadratic and cubic scalars respectively. The labels g and i are needed only when $h = 3$; then $g = 1$ denotes the linear and $g = 2$ the quadratic Γ_3 tensor; i labels the component.

SU(3) $\supset$ DT states in their final form are obtained by coupling centre states to orbit states with the help of DT Clebsch–Gordan coefficients. We denote an SU(3) $\supset$ DT state by $|{}^{pqabc}_{(g\Gamma_h, j\Gamma_k)\Gamma_l m}\rangle$; we suppress d because of (2.22) below. Here p, q are SU(3) representation labels and a, b are orbit labels. The values of r, s, t, needed for the orbit states, are generally given by equation (2.9), with $a, b \geq 0$ and hence $r \geq s \geq t$; the exception is an orbit with $a > 0, b = 0$; then we set $a = 0$ and $b = -a$ in (2.9), so that $t > r = s$. We write $p' = \Sigma$(positive exponents of r, s, t), $q' = -\Sigma$(negative exponents of r, s, t). Then

$$p = p' + 2c + 3d + m \qquad q = q' + 2c + 3d + m \tag{2.22}$$

where m is 0 if $i = 1$, 3 if $i = 2$, g if $h = 3$; c, d, g, Γ_h are as in the notation for the centre states in the preceding paragraph. j and Γ_i are as in the notation for orbit states, equations (2.11)–(2.19), with j needed only for Γ_3 in a hexagonal orbit. The centre irrep Γ_h and the orbit irrep Γ_k are coupled with DT Clebsch–Gordan coefficients to give all possible irrep Γ_l, component m.

Table 1. Clebsch–Gordan series for DT SU(3). The irrep Γ_i is denoted simply by i.

	1	2	3	4	5
1	1	2	3	4	5
2	2	1	3	5	4
3	3	3	$1 \oplus 2 \oplus 3$	$4 \oplus 5$	$4 \oplus 5$

The necessary DT Clebsch–Gordan series ($h = 1, 2, 3$) are shown in table 1. The relevant non-zero Clebsch–Gordan coefficients are given below

$$\Gamma_3 \times \Gamma_3 \to \Gamma_1 : \quad \left\langle \begin{matrix} 3 & 3 \\ 1 & 2 \end{matrix} \middle| \begin{matrix} 1 \\ 1 \end{matrix} \right\rangle = \left\langle \begin{matrix} 3 & 3 \\ 2 & 1 \end{matrix} \middle| \begin{matrix} 1 \\ 1 \end{matrix} \right\rangle = 2^{-1/2}$$

$$\Gamma_3 \times \Gamma_3 \to \Gamma_2 : \quad \left\langle \begin{matrix} 3 & 3 \\ 1 & 2 \end{matrix} \middle| \begin{matrix} 2 \\ 1 \end{matrix} \right\rangle = -\left\langle \begin{matrix} 3 & 3 \\ 2 & 1 \end{matrix} \middle| \begin{matrix} 2 \\ 1 \end{matrix} \right\rangle = 2^{-1/2}$$

$$\Gamma_3 \times \Gamma_3 \to \Gamma_3 : \quad \left\langle \begin{matrix} 3 & 3 \\ 1 & 1 \end{matrix} \middle| \begin{matrix} 3 \\ 2 \end{matrix} \right\rangle = \left\langle \begin{matrix} 3 & 3 \\ 2 & 2 \end{matrix} \middle| \begin{matrix} 3 \\ 1 \end{matrix} \right\rangle = 1$$

$$\Gamma_3 \times \Gamma_4 \to \Gamma_4 : \quad \left\langle \begin{matrix} 3 & 4 \\ 1 & 1 \end{matrix} \middle| \begin{matrix} 4 \\ 1 \end{matrix} \right\rangle = \left\langle \begin{matrix} 3 & 4 \\ 2 & 1 \end{matrix} \middle| \begin{matrix} 4 \\ 1 \end{matrix} \right\rangle = 2^{-1/2}$$

$$\omega \left\langle \begin{matrix} 3 & 4 \\ 1 & 2 \end{matrix} \middle| \begin{matrix} 4 \\ 2 \end{matrix} \right\rangle = \omega^2 \left\langle \begin{matrix} 3 & 4 \\ 2 & 2 \end{matrix} \middle| \begin{matrix} 4 \\ 2 \end{matrix} \right\rangle = 2^{-1/2}$$

$$\omega^2 \left\langle \begin{matrix} 3 & 4 \\ 1 & 3 \end{matrix} \middle| \begin{matrix} 4 \\ 3 \end{matrix} \right\rangle = \omega \left\langle \begin{matrix} 3 & 4 \\ 2 & 3 \end{matrix} \middle| \begin{matrix} 4 \\ 3 \end{matrix} \right\rangle = 2^{-1/2}$$

$$\Gamma_3 \times \Gamma_4 \to \Gamma_5 : \quad \left\langle \begin{matrix} 3 & 4 \\ 1 & 1 \end{matrix} \middle| \begin{matrix} 5 \\ 1 \end{matrix} \right\rangle = -\left\langle \begin{matrix} 3 & 4 \\ 2 & 1 \end{matrix} \middle| \begin{matrix} 5 \\ 1 \end{matrix} \right\rangle = 2^{-1/2}$$

$$\omega \left\langle \begin{matrix} 3 & 4 \\ 1 & 2 \end{matrix} \middle| \begin{matrix} 5 \\ 2 \end{matrix} \right\rangle = -\omega^2 \left\langle \begin{matrix} 3 & 4 \\ 2 & 2 \end{matrix} \middle| \begin{matrix} 5 \\ 2 \end{matrix} \right\rangle = 2^{-1/2}$$

$$\omega^2 \left\langle \begin{matrix} 3 & 4 \\ 1 & 3 \end{matrix} \middle| \begin{matrix} 5 \\ 3 \end{matrix} \right\rangle = -\omega \left\langle \begin{matrix} 3 & 4 \\ 2 & 3 \end{matrix} \middle| \begin{matrix} 5 \\ 3 \end{matrix} \right\rangle = 2^{-1/2}.$$

For $\Gamma_3 \times \Gamma_5 \to \Gamma_4$ and $\Gamma_3 \times \Gamma_5 \to \Gamma_5$ Clebsch–Gordan coefficients we remark that

$$\left\langle \begin{matrix} 3 & 5 \\ i & j \end{matrix} \middle| \begin{matrix} 4 \\ k \end{matrix} \right\rangle = \left\langle \begin{matrix} 3 & 4 \\ i & j \end{matrix} \middle| \begin{matrix} 5 \\ k \end{matrix} \right\rangle \quad \text{and} \quad \left\langle \begin{matrix} 3 & 5 \\ i & j \end{matrix} \middle| \begin{matrix} 5 \\ k \end{matrix} \right\rangle = \left\langle \begin{matrix} 3 & 4 \\ i & j \end{matrix} \middle| \begin{matrix} 4 \\ k \end{matrix} \right\rangle.$$

3. SO(5) ⊃ DT basis states

We construct basis states of the irrep (p,q) as polynomials of degree p in $\xi_1, \dots, \xi_5$ the basis states of the irrep (1,0) and of degree q in $\eta_1, \dots, \eta_4$ the basis states of the irrep (0,1). They are shown in figure 2.

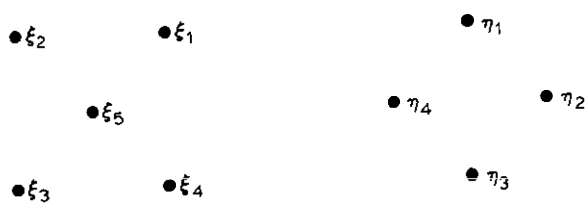

Figure 2. States of the fundamental irreps of SO(5).

As for SU(3) it is convenient to start with the SO(5) orbit generating function; it is given incorrectly by Michel *et al* (1988). We correct their misprint and also rewrite it in a form that has the centre orbits P and $(Q^2)_1, (Q^2)_2$ as common factors, a more convenient version for our use. It is then

$$F(P,Q;A,B) = [(1-P)(1-Q^2)^2(1-PA)]^{-1}\left[\frac{1+(P^2)(QB)+PQB+PQ^2}{(1-P^2)(1-P^2B^2)} + \frac{QB(1+PQB)}{(1-P^2B^2)(1-QB)} + \frac{Q^2A}{(1-QB)(1-Q^2A)}\right]. \tag{3.1}$$

The orbit generator implicitly tells us how to construct the states, but again we need guidance from the character generator (Patera and Sharp 1979) which we rewrite in a more convenient form

$$\begin{aligned} G(P,Q;A,B) = {} & [(1-P)(1-PA^{-1})(1-QAB^{-1})(1-QB)]^{-1} \\ & \times \{[(1-QB^{-1})(1-QA^{-1}B)]^{-1} + [(1-QA^{-1}B)(1-PA^{-1}B^2)]^{-1}PA^{-1}B^2 \\ & + [(1-PA^{-1}B^2)^{-1}(1-PA)^{-1}]^{-1}PA + [(1-PA)(1-PAB^{-2})]^{-1}PAB^{-2} \\ & + [(1-PAB^{-2})(1-QB^{-1})]PQAB^{-3}\}. \end{aligned} \tag{3.2}$$

The correspondence with the fundamental states is as follows: $PA\sim\xi_1$, $PA^{-1}B^2\sim\xi_2$, $PA^{-1}\sim\xi_3$, $PAB^{-2}\sim-\xi_4$, $P\sim\xi_5$, $QB\sim\eta_1$, $QAB^{-1}\sim\eta_2$, $QB^{-1}\sim\eta_3$, $QA^{-1}B\sim\eta_4$. The orbit generator (3.1) is the non-negative part, in A and B, of the character generator (3.2).

The pairs $\xi_2\xi_4$, $\xi_1\eta_3$, $\xi_2\eta_3$, $\xi_1\eta_4$, $\xi_4\eta_4$ never appear together in (3.2); for a consistent listing of states in an irrep (p,q), with p,q the degrees in the ξ and η variables respectively, we eliminate these combinations wherever they appear by setting to zero the unwanted states $\xi_1\xi_3-\xi_2\xi_4+\frac{1}{2}\xi_5^2$, $\xi_5\eta_1+2^{1/2}(\xi_1\eta_4-\xi_2\eta_2)$, $-\xi_5\eta_2+2^{1/2}(\xi_4\eta_1+\xi_1\eta_3)$, $-\xi_5\eta_4+2^{1/2}(\xi_3\eta_1+\xi_2\eta_3)$, $\xi_5\eta_3+2^{1/2}(\xi_3\eta_2-\xi_4\eta_4)$. The first belongs to the irrep (0,0), the other four to the irrep (0,1); a state containing any of them as a factor belongs to representations lower than its degree would indicate.

We now give the dominant states of the elementary orbits implied by the orbit generator (3.1). Eight appear in denominator factors, two appear only in a numerator. For the denominator elementary orbits, the correspondences are

$$\begin{array}{llll} P\sim\xi_5 & PA\sim\xi_1 & QB\sim\eta_1 & Q^2A\sim\eta_1\eta_2 \\ P^2\sim\xi_1\xi_3+\frac{1}{4}\xi_5^2 & P^2B^2\sim\xi_1\xi_2 & (Q^2)_1\sim\eta_1\eta_3 & (Q^2)_2\sim-\eta_2\eta_4 \end{array} \tag{3.3}$$

and for the numerator elementary orbits we have

$$PQB\sim\xi_2\eta_2-\tfrac{1}{4}\sqrt{2}\eta_1\xi_5 \qquad PQ^2\sim\xi_3\eta_1\eta_2+\tfrac{1}{4}\sqrt{2}\xi_5\eta_1\eta_3-\tfrac{1}{4}\sqrt{2}\xi_5\eta_2\eta_4. \tag{3.4}$$

For the time being we ignore the point elementary orbits $P, P^2, PQ^2, (Q^2)_1$ and $(Q^2)_2$. They can be coupled at the end and will not change orbit labels. Then there are three types of orbit, which are analysed below as cases 1,2 and 3.

The character table of DT SO(5) is given by Michel *et al* (1988) along with a representative element of each class. The irreps Γ_{1-4} and Γ_{6-9} are one dimensional; the other six are two dimensional. Standard matrices for the generating elements R_1 and R_2 can be reconstructed for each irrep from the states given below (under cases 1–3) with the help of their action on the fundamental states:

$$\begin{array}{llll} R_1\xi_1=\xi_2 & R_1\xi_2=-\xi_1 & R_1\xi_3=-\xi_4 & R_1\xi_4=\xi_3 \\ R_1\xi_5=\xi_5 & R_1\eta_1=\eta_1 & R_1\eta_2=\eta_4 & R_1\eta_3=\eta_3 \\ R_1\eta_4=-\eta_2 & R_2\xi_1=\xi_1 & R_2\xi_2=\xi_4 & R_2\xi_3=\xi_3 \\ R_2\xi_4=\xi_2 & R_2\xi_5=-\xi_5 & R_2\eta_1=\eta_2 & R_2\eta_2=-\eta_1. \\ R_2\eta_3=-\eta_4 & R_4\eta_4=\eta_3. & & \end{array} \tag{3.5}$$

The DT content of each type of orbit is determined with the help of the character table. The states themselves are then found straightforwardly. The notation is $|i,\Gamma_j,k\rangle$ where Γ_j is the irrep (we adopt the labelling of Michel *et al* (1988)) k is the component label, where necessary, and i labels the copy, when there is more than one in the orbit. We assume the exponents x,y,z all greater than zero, hence octagonal orbits. Afterwards we mention what happens in degenerate (square) orbits.

Case 1. The dominant state is $\xi_1^{x+y}\xi_2^y\eta_1^z$. Let (a,b,c,d,e,f,g,h) denote $a\xi_1^{x+y}\xi_2^y\eta_1^z+b\xi_1^y\xi_2^{x+y}\eta_1^z+c\xi_2^{x+y}\xi_3^y\eta_4^z+d\xi_2^y\xi_3^{x+y}\eta_4^z+e\xi_3^{x+y}\xi_4^y\eta_3^z+f\xi_3^y\xi_4^{x+y}\eta_3^z+g\xi_1^y\xi_4^{x+y}\eta_2^z+h\xi_1^{x+y}\xi_4^y\eta_2^z$. Then $(\alpha=(-1)^y)$

x even, z even

$$
\begin{array}{ll}
|\Gamma_1\rangle = (1,\alpha,1,\alpha,\alpha,1,\alpha,1) & |\Gamma_2\rangle = (1,\alpha,-1,-\alpha,\alpha,1,-\alpha,-1) \\
|\Gamma_3\rangle = (1,-\alpha,-1,\alpha,\alpha,-1,-\alpha,1) & |\Gamma_4\rangle = (1,-\alpha,1,-\alpha,\alpha,-1,\alpha,-1) \\
|1,\Gamma_5,1\rangle = (1,\alpha,0,0,-\alpha,-1,0,0) & |1,\Gamma_5,2\rangle = (0,0,-1,-\alpha,0,0,\alpha,1) \\
|2,\Gamma_5,1\rangle = (0,0,1,-\alpha,0,0,-\alpha,1) & |2,\Gamma_5,2\rangle = (1,-\alpha,0,0,-\alpha,1,0,0).
\end{array}
$$

x odd, z even

$$
\begin{array}{ll}
|\Gamma_6\rangle = (1,-i\alpha,-i,-\alpha,-\alpha,-i,-i\alpha,1) & |\Gamma_7\rangle = (1,i\alpha,i,-\alpha,-\alpha,i,i\alpha,1) \\
|\Gamma_8\rangle = (1,-i\alpha,i,\alpha,-\alpha,-i,i\alpha,-1) & |\Gamma_9\rangle = (1,i\alpha,-i,\alpha,-\alpha,i,-i\alpha,-1) \\
|1,\Gamma_{14},1\rangle = (1,-i\alpha,0,0,\alpha,i,0,0) & |1,\Gamma_{14},2\rangle = (0,0,i,\alpha,0,0,-i\alpha,1) \\
|2,\Gamma_{14},1\rangle = (0,0,-i,\alpha,0,0,i\alpha,1) & |2,\Gamma_{14},2\rangle = (1,i\alpha,0,0,\alpha,-i,0,0).
\end{array}
$$

x even, z odd

$$
\begin{array}{ll}
|\Gamma_{10}1\rangle = (0,0,1,\alpha,0,0,i\alpha,i) & |\Gamma_{10}2\rangle = (-i,-i\alpha,0,0,\alpha,1,0,0) \\
|\Gamma_{11}1\rangle = (0,0,1,\alpha,0,0,-i\alpha,-i) & |\Gamma_{11}2\rangle = (i,i\alpha,0,0,\alpha,1,0,0) \\
|\Gamma_{12},1\rangle = (0,0,1,-\alpha,0,0,i\alpha,-i) & |\Gamma_{12},2\rangle = (i,-i\alpha,0,0,-\alpha,1,0,0) \\
|\Gamma_{13},1\rangle = (0,0,1,-\alpha,0,0,i\alpha,i) & |\Gamma_{13},2\rangle = (-i,i\alpha,0,0,-\alpha,1,0,0).
\end{array}
$$

x odd, z odd

$$
\begin{array}{ll}
|\Gamma_{10},1\rangle = (1,-i\alpha,0,0,i\alpha,-1,0,0) & |\Gamma_{10},2\rangle = (0,0,1,-i\alpha,0,0,-i\alpha,1) \\
|\Gamma_{11},1\rangle = (1,i\alpha,0,0,-i\alpha,-1,0,0) & |\Gamma_{11},2\rangle = (0,0,1,i\alpha,0,0,i\alpha,1) \\
|\Gamma_{12},1\rangle = (1,i\alpha,0,0,i\alpha,1,0,0) & |\Gamma_{12},2\rangle = (0,0,-1,-i\alpha,0,0,i\alpha,i) \\
|\Gamma_{13},1\rangle = (1,-i\alpha,0,0,-i\alpha,1,0,0) & |\Gamma_{13},2\rangle = (0,0,-1,i\alpha,0,0,-i\alpha,1).
\end{array}
$$

Case 2. The dominant state is $\xi_1^x\eta_1^{y+z}\eta_2^z$, $z > 0$. Let (a,b,c,d,e,f,g,h) denote $a\xi_1^x\eta_1^{y+z}\eta_2^z + b\xi_2^x\eta_1^{y+z}\eta_4^z + c\xi_2^x\eta_1^z\eta_4^{y+z} + d\xi_3^x\eta_3^z\eta_4^{y+z} + e\xi_3^x\eta_3^{y+z}\eta_4^z + f\xi_4^x\eta_2^z\eta_3^{y+z} + g\xi_4^x\eta_2^{y+z}\eta_3^z + h\xi_1^x\eta_1^z\eta_2^{y+z}$. Then

$x+z$ even, y odd, $\alpha = (-1)^x = (-1)^z$

$$
\begin{array}{ll}
|\Gamma_{10},1\rangle = (0,0,1,\alpha,0,0,i\alpha,i) & |\Gamma_{10},2\rangle = (-i\alpha,-i\alpha,0,0,1,1,0,0) \\
|\Gamma_{11},1\rangle = (0,0,1,\alpha,0,0,-i\alpha,-i) & |\Gamma_{11},2\rangle = (i\alpha,i\alpha,0,0,1,1,0,0) \\
|\Gamma_{12},1\rangle = (0,0,1,-\alpha,0,0,i\alpha,-i) & |\Gamma_{12},2\rangle = (i\alpha,-i\alpha,0,0,-1,1,0,0) \\
|\Gamma_{13},1\rangle = (0,0,1,-\alpha,0,0,-i\alpha,i) & |\Gamma_{13},2\rangle = (-i\alpha,i\alpha,0,0,-1,-1,0,0).
\end{array}
$$

$x+z$ odd, y odd, $\alpha = (-1)^x = -(-1)^z$

$$
\begin{array}{ll}
|\Gamma_{10},1\rangle = (1,-i,0,0,-i\alpha,\alpha,0,0) & |\Gamma_{10},2\rangle = (0,0,-\alpha,-i,0,0,-i,-\alpha) \\
|\Gamma_{11},1\rangle = (1,i,0,0,i\alpha,\alpha,0,0) & |\Gamma_{11},2\rangle = (0,0,-\alpha,i,0,0,i,-\alpha) \\
|\Gamma_{12},1\rangle = (1,i,0,0,-i\alpha,-\alpha,0,0) & |\Gamma_{12},2\rangle = (0,0,\alpha,-i,0,0,i,-\alpha), \\
|\Gamma_{13},1\rangle = (1,-i,0,0,i\alpha,-\alpha,0,0) & |\Gamma_{13},2\rangle = (0,0,\alpha,i,0,0,-i,-\alpha).
\end{array}
$$

$x+z$ even, y even, $\alpha = (-1)^x = (-1)^z$

$$|\Gamma_1\rangle = (1,1,\alpha,1,\alpha,\alpha,1,\alpha) \qquad |\Gamma_2\rangle = (1,1,-\alpha,-1,\alpha,\alpha,-1,-\alpha)$$
$$|\Gamma_3\rangle = (1,-1,-\alpha,1,\alpha,-\alpha,-1,\alpha) \qquad |\Gamma_4\rangle = (1,-1,\alpha,-1,\alpha,-\alpha,1,-\alpha)$$
$$|1,\Gamma_5,1\rangle = (1,1,0,0,-\alpha,-\alpha,0,0) \qquad |1,\Gamma_5,2\rangle = (0,0,-\alpha,-1,0,0,1,\alpha)$$
$$|2,\Gamma_5,1\rangle = (0,0,1,-\alpha,0,0,-\alpha,1) \qquad |2,\Gamma_5,2\rangle = (\alpha,-\alpha,0,0,-1,1,0,0).$$

$x+z$ odd, y even, $\alpha = (-1)^x = -(-1)^z$

$$|\Gamma_6\rangle = (1,-i,i\alpha,-1,\alpha,i\alpha,-i,-\alpha) \qquad |\Gamma_7\rangle = (1,i,-i\alpha,-1,\alpha,-i\alpha,i,-\alpha)$$
$$|\Gamma_8\rangle = (1,-i,-i\alpha,1,\alpha,i\alpha,i,\alpha) \qquad |\Gamma_9\rangle = (1,i,i\alpha,1,\alpha,-i\alpha,-i,\alpha)$$
$$|1,\Gamma_{14},1\rangle = (1,-i,0,0,-\alpha,-i\alpha,0,0) \qquad |1,\Gamma_{14}2\rangle = (0,0,-i\alpha,1,0,0,-i,-\alpha)$$
$$|2,\Gamma_{14},1\rangle = (0,0,1,-i\alpha,0,0,\alpha,i) \qquad |2,\Gamma_{14},2\rangle = (-i\alpha,\alpha,0,0,i,1,0,0).$$

Case 3. The dominant state is $N_1\xi^{x+y}\xi_2^y\eta_1^z$

Here $N_1 = \xi_2\eta_2 - \frac{1}{4}\sqrt{2}\,\xi_5\eta_1$, $N_2 = \xi_4\eta_1 - \frac{1}{4}\sqrt{2}\,\xi_5\eta_2$, $N_3 = -\xi_3\eta_2 - \frac{1}{4}\sqrt{2}\,\xi_5\eta_3$, $N_4 = \xi_3\eta_1 - \frac{1}{4}\sqrt{2}\,\xi_5\eta_4$.

Let (a,b,c,d,e,f,g,h) denote $aN_1\xi_1^{x+y}\xi_2^y\eta_1^z + bN_1\xi_1^y\xi_2^{x+y}\eta_1^z + cN_4\xi_2^{x+y}\xi_3^y\eta_4^z + dN_4\xi_2^y\xi_3^{x+y}\eta_4^z + eN_3\xi_3^{x+y}\xi_4^y\eta_3^z + fN_3\xi_3^y\xi_4^{x+y}\eta_3^z + gN_2\xi_1^y\xi_4^{x+y}\eta_2^z + hN_2\xi_1^{x+y}\xi_4^y\eta_2^z$

Then ($\alpha = (-1)^y$)

x even, z even

$$|\Gamma_{10},1\rangle = (0,0,1,-\alpha,0,0,-i\alpha,i) \qquad |\Gamma_{10},2\rangle = (i,-i\alpha,0,0,\alpha,-1,0,0)$$
$$|\Gamma_{11},1\rangle = (0,0,1,-\alpha,0,0,i\alpha,-i) \qquad |\Gamma_{11},2\rangle = (-i,i\alpha,0,0,\alpha,-1,0,0)$$
$$|\Gamma_{12},1\rangle = (0,0,1,\alpha,0,0,-i\alpha,-i) \qquad |\Gamma_{12},2\rangle = (-i,-i\alpha,0,0,-\alpha,-1,0,0)$$
$$|\Gamma_{13},1\rangle = (0,0,1,\alpha,0,0,i\alpha,i) \qquad |\Gamma_{13},2\rangle = (i,i\alpha,0,0,-\alpha,-1,0,0).$$

x odd, z even

$$|\Gamma_{10},1\rangle = (1,i\alpha,0,0,-i\alpha,-1,0,0) \qquad |\Gamma_{10},2\rangle = (0,0,-1,-i\alpha,0,0,-i\alpha,-1)$$
$$|\Gamma_{11},1\rangle = (1,-i\alpha,0,0,i\alpha,-1,0,0) \qquad |\Gamma_{11},2\rangle = (0,0,-1,i\alpha,0,0,i\alpha,-1)$$
$$|\Gamma_{12},1\rangle = (1,-i\alpha,0,0,-i\alpha,1,0,0) \qquad |\Gamma_{12},2\rangle = (0,0,1,-i\alpha,0,0,i\alpha,-1)$$
$$|\Gamma_{13},1\rangle = (1,i\alpha,0,0,i\alpha,1,0,0) \qquad |\Gamma_{13},1\rangle = (0,0,1,i\alpha,0,0,-i\alpha,-1).$$

x even, z odd

$$|\Gamma_1\rangle = (1,-\alpha,-1,\alpha,-\alpha,1,\alpha,-1) \qquad |\Gamma_2\rangle = (1,-\alpha,1,-\alpha,-\alpha,1,-\alpha,1)$$
$$|\Gamma_3\rangle = (1,\alpha,1,\alpha,-\alpha,-1,-\alpha,-1) \qquad |\Gamma_4\rangle = (1,\alpha,-1,-\alpha,-\alpha,-1,\alpha,1)$$
$$|1,\Gamma_5,1\rangle = (1,-\alpha,0,0,\alpha,-1,0,0) \qquad |1,\Gamma_5,2\rangle = (0,0,1,-\alpha,0,0,\alpha-1)$$
$$|2,\Gamma_5,1\rangle = (0,0,1,\alpha,0,0,\alpha,1) \qquad |2,\Gamma_5,2\rangle = (-1,-\alpha,0,0,-\alpha,-1,0,0).$$

x odd, z odd

$$|\Gamma_6\rangle = (1,i\alpha,i,-\alpha,\alpha,-i,-i\alpha,-1) \qquad |\Gamma_7\rangle = (1,-i\alpha,-i,-\alpha,\alpha,i,\alpha,-1)$$
$$|\Gamma_8\rangle = (1,i\alpha,-i,\alpha,\alpha,-i,i\alpha,1) \qquad |\Gamma_9\rangle = (1,-i\alpha,i,\alpha,\alpha,i,-i\alpha,1)$$
$$|1,\Gamma_{14},1\rangle = (1,i\alpha,0,0,-\alpha,i,0,0) \qquad |1,\Gamma_{14}2\rangle = (0,0,-i,\alpha,0,0,-i\alpha,-1)$$
$$|2,\Gamma_{14},1\rangle = (0,0,1,-i\alpha,0,0,\alpha,i) \qquad |2,\Gamma_{14},2\rangle = (-i,-\alpha,0,0,i\alpha,-1,0,0).$$

The states of degenerate, or square, orbits (a,0) or (0,b) are obtained as special cases of the generic, or octagonal, orbit states above.

For case 1, with $x = 0$, z even we find that the states $|\Gamma_1\rangle, |\Gamma_2\rangle, |1, \Gamma_5, i\rangle$ vanish for y odd while $|\Gamma_3\rangle, |\Gamma_4\rangle, |2\Gamma_5, i\rangle$ vanish for y even; for $x = 0$, z odd $|\Gamma_{10}, i\rangle$ and $|\Gamma_{11}, i\rangle$ vanish for y odd while $|\Gamma_{12}, i\rangle$ and $|\Gamma_{13}, i\rangle$ vanish for y even; for $y = z = 0, x$ even $|\Gamma_2\rangle$ and $|\Gamma_4\rangle$ vanish and $|1, \Gamma_5, i\rangle = |2, \Gamma_5, i\rangle$; for $y = z = 0, x$ odd, $|\Gamma_8\rangle$ and $|\Gamma_9\rangle$ vanish while $|1, \Gamma_{14}, i\rangle = |2, \Gamma_{14}, i\rangle$.

For case 2, with $y = 0$, $x + z$ even, $|\Gamma_1\rangle$ and $|\Gamma_3\rangle$ vanish for x and z odd while $|\Gamma_2\rangle$ and $|\Gamma_4\rangle$ vanish for x and z even; $|1, \Gamma_5, i\rangle = |2, \Gamma_5, i\rangle$ for x and z even or odd. For $x = 0$, $x + z$ odd, $|\Gamma_6\rangle$ and $|\Gamma_7\rangle$ vanish for x even while $|\Gamma_8\rangle$ and $|\Gamma_9\rangle$ vanish for x odd; $|2, \Gamma_{14}, i\rangle = i|1, \Gamma_{14}, i\rangle$ for x even or odd. For $x = z = 0$, $|\Gamma_{12}, i\rangle$ and $|\Gamma_{13}, i\rangle$ vanish for y odd, while $|\Gamma_3\rangle$, $|\Gamma_4\rangle$ and $|2, \Gamma_5, i\rangle$ vanish for y even.

For case 3 with $x = 0$, z even, $|\Gamma_{10}, i\rangle$ and $|\Gamma_{11}, i\rangle$ vanish for y even while $|\Gamma_{12}, i\rangle$ and $|\Gamma_{13}, i\rangle$ vanish for y odd. With $x = 0$, z odd, $|\Gamma_1\rangle$, $|\Gamma_2\rangle$ and $|1, \Gamma_5, i\rangle$ vanish for y even while $|\Gamma_3\rangle$, $|\Gamma_4\rangle$ and $|2, \Gamma_5, i\rangle$ vanish for y odd.

Finally we must couple the point orbits to the states already constructed. The orbit-generating function (3.1) tells us which point orbits may be combined with which of the states listed under cases 1–3 above. Thus, PQ^2 may be multiplied by the states of case 1 for which $z = 0$; any power of P^2 may be multiplied by the states of case 1 for which $z = 0$ or 1, by the states of case 3 for which $z = 0$ and by any state containing PQ^2 as a factor. Since the point orbits P and $(Q^2)_1$, $(Q^2)_2$ are in common denominator factors, any powers of them can be coupled to any other state. All the point orbits, with the exception of $(Q^2)_1$, $(Q^2)_2$, belong to one-dimensional representations of DT, so coupling them is trivial.

$(Q^2)_1$, $(Q^2)_2$ *span the two-dimensional irrep* Γ_5 . We follow Patera *et al* (1978) in constructing DT tensors whose components are polynomials in $(Q^2)_1$, $(Q^2)_2$. Their construction tells us that there are Γ_1 tensors (scalars) of degrees 2 and 4, a Γ_2 tensor and a Γ_3 tensor of degree 2, a Γ_4 tensor of degree 4, and Γ_5 tensors of degrees 1 and 3. Any of these can be multiplied by powers of the scalars to obtain higher-degree tensors. It is trivial, and left to the reader, to find the polynomial tensors and the DT Clebsch–Gordan coefficients needed to couple the Γ_5 irreps to two-dimensional irreps in the orbit states already analysed (or see de Guise (1989)).

4. Concluding remarks

In this section we make a few comments about applications of our basis states, in particular their potential use in the calculation of generator matrix elements.

Our states of degrees p, q, respectively in the variables of the two fundamental irreps are in one-to-one relation to the actual (p, q) states in a DT basis, but as given they contain admixtures of states (of the same DT irrep) belonging to lower representations of SU(3) or SO(5). This turns out not to be a problem for the calculation of generator matrix elements. In the case of SU(3) one has formally only to replace η_i and η_i^* by the traceless variables η_i' and $\eta_i'^*$ described by Patera *et al* (1989). The primed and unprimed variables transform in the same way under finite SU(3) transformations, or under the action of SU(3) generators. No such traceless variables are available for SO(5) (but see Lohe and Hurst (1971) for a redefinition of the ξ variables which eliminates the presence of the scalar quadratic in them); however, the same effect is achieved by operating on each state with an operator P which is an instruction to retain only the irrep (p, q) part of its operand. Since P commutes with SO(5) generators (or finite transformations) one can ignore the role of P and work with the states as they stand.

The generators should of course be organized into DT tensors, transforming by Γ_3, Γ_4, Γ_5 for SU(3) and by Γ_1, Γ_2, Γ_8, Γ_9, $2\Gamma_5$, Γ_{14} for SO(5) and their matrix elements presented as reduced matrix elements multiplied by DT Clebsch-Gordan coefficients. Because of the non-orthonormality of the basis states the matrix elements E_{ij} of a generator E should be defined by

$$E|j\rangle = \sum_i |i\rangle E_{ij}$$

and not through a scalar product. They can still be multiplied as matrices, and functions of them, for example a Hamiltonian that lies in the enveloping algebra, can be diagonalized conventionally and eigenstates found as linear combinations of basis states.

The methods of this paper could be extended to obtain DT basis states of higher groups; but the work becomes rapidly more tedious because of the increasing complexity of the orbit-generating function in particular.

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